

Characterizing the Adjacency Matrix of the Skeleton of the Strict Hypergraph of Layered Generalized Crowns

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Abstract

In 1974, Trotter provided a formula to compute the order dimension of the generalized crown \mathbb{S}_n^k , see [3]. In 2014, Garcia and Silva defined a layering of generalized crowns and provided a formula for the order dimension of the ℓ -layered generalized crown, see [1]. In this paper, we characterize the adjacency matrix of the skeleton of the strict hypergraph of the ℓ -layered generalized crown.

Introduction

In [3], Trotter defined the generalized crown \mathbb{S}_n^k as a height two poset with maximal elements b_1, \dots, b_{n+k} and minimal elements a_1, \dots, a_{n+k} . Each b_i is incomparable with $a_i, a_{i+1}, \dots, a_{i+k}$, and comparable over the remaining $n-1$ elements. It is assumed that any b_i is incomparable to b_j for all $j \neq i$. For an illustration see Figure 1, which provides the Hasse diagram of the generalized crown \mathbb{S}_4^2 .

Garcia and Silva defined in [1] the operation of layering posets. The layering “... produces a larger poset from two compatible posets by gluing one poset above the other in a well defined way.” For an illustration see Figure 5, which provides the diagram of $\mathbb{S}_3^2 \rtimes \mathbb{S}_3^2$.

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In this paper, we characterize the adjacency matrix of the skeleton of the strict hypergraph associated to the ℓ -layered generalized crown $\rtimes_{\ell} \mathbb{S}_n^k$. We denote this matrix by $\mathcal{A}_n^k(\ell)$. The computation of $\mathcal{A}_n^k(\ell)$ depends on the values of n, k , and ℓ .

In Section 1, we provide background material and definitions. Section 2 computes $\mathcal{A}_n^k(1) := \mathcal{A}_n^k$ for generalized crowns. In Section 3, we determine $\mathcal{A}_n^k(\ell)$ for the case where $n > k + 3$ and $\ell \geq 1$. The case where $3 \leq n < k + 3$ and $\ell \geq 1$ is covered in Section 4. These sections provide a complete description of the matrix $\mathcal{A}_n^k(\ell)$ for all possible values for all ℓ -layered generalized crowns.

1 Background

The notation and definitions in this paper are consistent with those used in [1].

Definition 1.1. *Let X be a set, called the ground set, and let $P \subset X \times X$ be a partial ordering on X with the following binary relations on X :*

- (i) *reflexive: for all $a \in X$ we have $a \leq a$;*
- (ii) *antisymmetry: if $a \leq b$ and $b \leq a$, then $a = b$; and*
- (iii) *transitivity: if $a \leq b$ and $b \leq c$, then $a \leq c$.*

Then the pair $\mathbb{P} = (X, P)$ is called poset or partially ordered set.

All ground sets X considered in this paper are finite.

Example 1.1. *Let X be a finite set and let the ground set of our poset be the power set $\wp(X)$. Consider a partial ordering on this set by containment of subsets. Then $\mathbb{P} = (\wp(X), \subseteq)$ forms a partially ordered set, i.e. a poset.*

Notation 1.1. *Let X be a finite ground set and let $a, b \in X$. We write $b \parallel a$ to denote that b is incomparable with a . We write $b > a$ or (a, b) to denote that b lies over a in the partial ordering.*

Our paper focuses on a special family of posets called generalized crowns. This definition was originally introduced by Trotter; see [2].

Definition 1.2. *Let $n, k \in \mathbb{N}$ with $n \geq 3$ and $k \geq 0$. Then the generalized crown, denoted \mathbb{S}_n^k , is a height 2 poset with $\min(\mathbb{S}_n^k) = \{a_1, \dots, a_{k+n}\}$ and $\max(\mathbb{S}_n^k) = \{b_1, \dots, b_{k+n}\}$, where*

- (1) $b_i \parallel a_i, a_{i+1}, \dots, a_{i+k}$, corresponding to $k + 1$ “misses,”
- (2) $b_i > a_{i+k+1}, a_{i+k+2}, \dots, a_{i-1}$, corresponding to $n - 1$ “hits.”

Example 1.2. The following picture demonstrates how the terminology “crown” evolved.

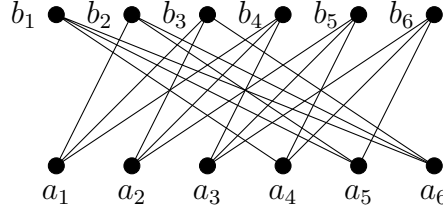


Figure 1: \mathbb{S}_4^2

Definition 1.3. Elements in the top row of the crown are called maximal elements and the set of all such elements is denoted by $B = \max(\mathbb{S}_n^k)$. Elements in the bottom row of the crown are called minimal elements and the set of minimal elements is denoted by $A = \min(\mathbb{S}_n^k)$. Thus we have $X = A \cup B$ for the ground set of the crown \mathbb{S}_n^k .

We identify a_i with $a_{i-(n+k)}$ and b_i with $b_{i-(n+k)}$ whenever $i > n+k$. This indexing scheme is called cyclic indexing.

Definition 1.4. Let $\mathbb{P} = (X, P)$ be a poset and let $x \in X$. We say the strict downset of x is the set

$$D_{\mathbb{P}}(x) = \{y \in X : y <_P x\}$$

and the strict upset of x is the set

$$U_{\mathbb{P}}(x) = \{y \in X : x <_P y\}.$$

Definition 1.5. Let $\mathbb{P} = (X, P)$ be a poset and let $x \in X$. We say that x is minimal if there does not exist $y \in X$ such that $y <_P x$. We say that x is maximal if there does not exist $y \in X$ such that $x <_P y$.

When the poset \mathbb{P} is understood we drop the subscript notation.

Definition 1.6. For a maximal element $b \in \mathbb{S}_n^k$, the set of all minimal elements of \mathbb{S}_n^k that are incomparable with b is denoted by

$$\text{Inc}(b) = \{a \in A : b \parallel a\}.$$

For a minimal element $a \in \mathbb{S}_n^k$, the set of all maximal elements of \mathbb{S}_n^k that are incomparable to a is denoted by

$$\text{Inc}(a) = \{b \in B : b \parallel a\}.$$

The set of all incomparable pairs of \mathbb{S}_n^k is denoted by

$$\text{Inc}(\mathbb{S}_n^k) = \{(x, y) \in \mathbb{S}_n^k \times \mathbb{S}_n^k : x \parallel y\}.$$

Definition 1.7. Let $\mathbb{P} = (X, P)$ be a poset and let $x, y \in X$. We call (x, y) a critical pair if the following conditions hold:

- (i) $x||y$;
- (ii) $D(x) \subset D(y)$; and
- (iii) $U(y) \subset U(x)$.

We let $\text{Crit}(\mathbb{P})$ denote the set of all critical pairs of \mathbb{P} .

The next few definitions set the ground work for obtaining results concerning hypergraphs of generalized crowns.

Definition 1.8. An alternating cycle in a poset \mathbb{P} is a sequence $\{(x_i, y_i) : 1 \leq i \leq k\}$ of ordered pairs from $\text{Inc}(\mathbb{P})$, where $y_i \leq x_{i+1}$ in \mathbb{P} (cyclically) for $i = 1, 2, \dots, k$.

An alternating cycle is strict if $y_i \leq x_j$ in $\mathbb{P} \iff j = i + 1$ (cyclically) for $i, j = 1, 2, \dots, k$.

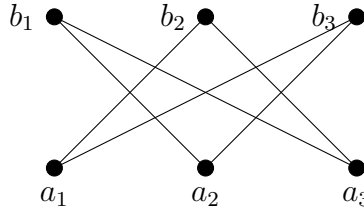


Figure 2: \mathbb{S}_3^0

Example 1.3. Given the poset \mathbb{S}_3^0 , we have $\text{Crit} \mathbb{S}_3^0 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$. The inequality in the alternating cycle definition implies that we need to consider duals of the critical pairs when determining which sets form an alternating cycle. Note that the duals of critical pairs are in the set of incomparable pairs. In the case of \mathbb{S}_3^0 , any combination of two or more duals of the critical pairs (b_i, a_i) forms an alternating cycle. For example, $\{(b_2, a_2), (b_1, a_1)\}$ is a strict alternating cycle while $\{(b_3, a_3), (b_2, a_2), (b_1, a_1)\}$ is an alternating cycle which is not strict.

Definition 1.9. A hypergraph $H = (V, E)$ is a set V of vertices along with a set E of edges which are subsets of V with size ≥ 2 . If an edge has size 2, it is called a graph edge. If an edge has size ≥ 3 , it is called a hyperedge. If a graph has only graph edges, then it is simply called a graph.

Definition 1.10. Given a hypergraph $H = (V, E_H)$, define the skeleton of H , denoted $Sk[H]$, to be (V, E_G) , where $E_G = \{e \in E_H : |e| = 2\}$.

Definition 1.11. Given a poset \mathbb{P} , the strict hypergraph of critical pairs of \mathbb{P} , or strict hypergraph, denoted $K^s(\mathbb{P})$, is the hypergraph $(\text{Crit}(\mathbb{P}), F)$, where F consists of subsets of $\text{Crit}(\mathbb{P})$ whose duals form strict alternating cycles.

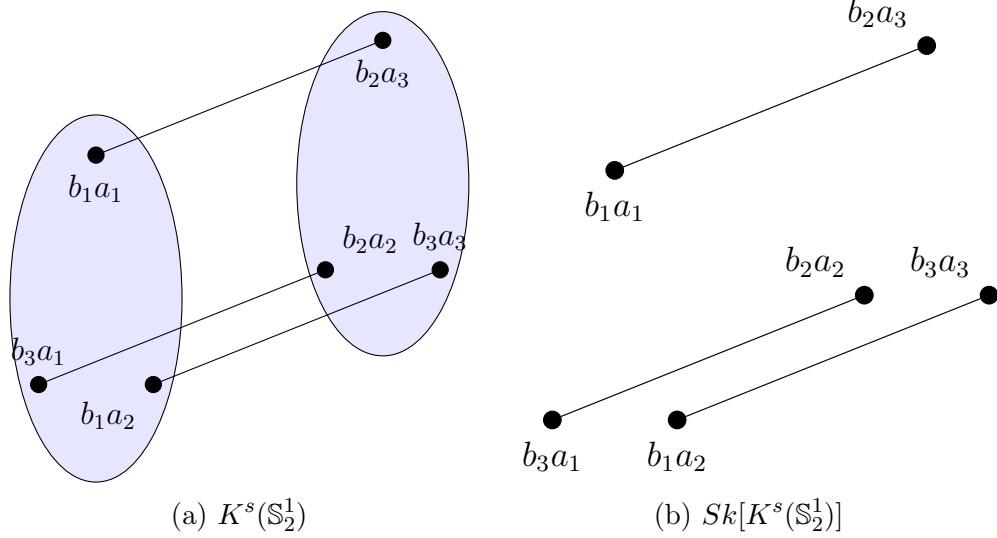


Figure 3: The strict hypergraph and the skeleton of the strict hypergraph of \mathbb{S}_2^1 .

Example 1.4. Figure (3a) above shows the hypergraph of the critical pairs of \mathbb{S}_2^1 . Figure (3b) shows the skeleton of the strict hypergraph of critical pairs of \mathbb{S}_2^1 , denoted $Sk[K^s(\mathbb{S}_2^1)]$. The critical pairs are $\text{Crit } \mathbb{S}_2^1 = \{(a_1, b_1), (a_1, b_3), (a_2, b_2), (a_2, b_1), (a_3, b_3), (a_3, b_2)\}$ and are the vertices of the hypergraph. The edges are $\{(b_1, a_1), (b_2, a_3)\}$, $\{(b_3, a_1), (b_2, a_2)\}$, $\{(b_1, a_2), (b_3, a_3)\}$, $\{(b_1, a_1), (b_2, a_2), (b_3, a_3)\}$, and $\{(b_3, a_1), (b_2, a_3), (b_1, a_2)\}$ where the first three cycles are graph edges.

Definition 1.12. Given a graph $G = (V, E)$ with $v = \{1, 2, \dots, n\}$, the adjacency matrix of G , denoted $M(G)$, is defined by $m_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$

2 Adjacency matrix of $Sk[K^s(\mathbb{S}_n^k)]$

Notation is crucial when working with adjacency matrices. The notation below describes how we label the rows and columns of the matrix. This notation is extended in the next section to deal with layered crowns.

Notation 2.1. For $n \geq 3$ and $k \geq 0$, let \mathcal{A}_n^k be the adjacency matrix of $Sk[K^s(\mathbb{S}_n^k)]$. Fix $1 \leq i, j \leq n + k$ and let $A_{i,j}$ denote the $(k + 1) \times (k + 1)$ submatrix whose rows are given by the $k + 1$ critical pairs of the form (a_u, b_i) and columns are indexed by the $k + 1$ critical pairs of the form (a_v, b_j) . Then

$$\mathcal{A}_n^k = [A_{i,j}]_{1 \leq i, j \leq n+k}.$$

For fixed $1 \leq i, j \leq n + k$ we let $m_{u,v}$ denote the (u, v) -entry of matrix $A_{i,j}$. Notice that u ranges from i to $k + i$, where the order is fixed and all terms are taken modulo $n + k$; see

definition 1.3 to recall cyclic indexing. Similarly, v ranges from j to $j + k$, where the order is fixed and the terms are taken modulo $n + k$. We denote these ranges by writing

$$u \in [i, i + 1, \dots, k + i] \pmod{(n+k)}$$

and

$$v \in [j, j + 1, \dots, k + j] \pmod{(n+k)}.$$

Notation 2.2. Recall that there are $(k+1)(n+k)$ critical pairs arising from the $k+1$ misses for each of the $n+k$ nodes. These critical pairs are of the form

$$(a_1, b_1), \dots, (a_{k+1}, b_1), (a_2, b_2), (a_3, b_2), \dots, (a_{2+k}, b_2), \dots, (a_{n+k}, b_{n+k}), \dots, (a_k, b_{n+k}).$$

In the above listing, we order the critical pairs by lexicographical order on their dual. This labeling is also used in the $(k+1)(n+k)$ rows (and by symmetry columns) of the matrix \mathcal{A}_n^k .

Example 2.1. Consider the graph for \mathbb{S}_6^1 below:

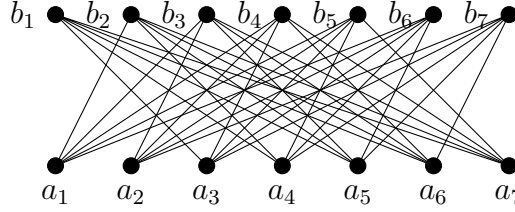


Figure 4: \mathbb{S}_6^1

The critical pairs are given as follows:

$$(a_1, b_1), (a_2, b_1), (a_2, b_2), (a_3, b_2), (a_3, b_3), (a_4, b_3), (a_4, b_4), \\ (a_5, b_4), (a_5, b_5), (a_6, b_5), (a_6, b_6), (a_7, b_6), (a_7, b_7), (a_1, b_7).$$

Examining which duals of critical pairs form strict alternating cycles verifies that the adjacency matrix of $\mathcal{A}_6^1 = Sk[K^s(\mathbb{S}_6^1)]$ is as follows:

	(a_1, b_1)	(a_2, b_1)	(a_2, b_2)	(a_3, b_2)	(a_3, b_3)	(a_4, b_3)	(a_4, b_4)	(a_5, b_4)	(a_5, b_5)	(a_6, b_5)	(a_6, b_6)	(a_7, b_6)	(a_7, b_7)	(a_1, b_7)
(a_1, b_1)	0	0	0	1	1	1	1	1	1	1	1	1	0	0
(a_2, b_1)	0	0	0	0	1	1	1	1	1	1	1	1	1	0
(a_2, b_2)	0	0	0	0	0	1	1	1	1	1	1	1	1	1
(a_3, b_2)	1	0	0	0	0	0	1	1	1	1	1	1	1	1
(a_3, b_3)	1	1	0	0	0	0	0	1	1	1	1	1	1	1
(a_4, b_3)	1	1	1	0	0	0	0	0	1	1	1	1	1	1
(a_4, b_4)	1	1	1	1	0	0	0	0	0	1	1	1	1	1
(a_5, b_4)	1	1	1	1	1	0	0	0	0	0	1	1	1	1
(a_5, b_5)	1	1	1	1	1	1	0	0	0	0	0	1	1	1
(a_6, b_5)	1	1	1	1	1	1	1	0	0	0	0	0	1	1
(a_6, b_6)	1	1	1	1	1	1	1	1	0	0	0	0	0	1
(a_7, b_6)	1	1	1	1	1	1	1	1	1	0	0	0	0	0
(a_7, b_7)	0	1	1	1	1	1	1	1	1	1	0	0	0	0
(a_1, b_7)	0	0	1	1	1	1	1	1	1	1	1	0	0	0

Because the edges of $Sk[K^s(\mathbb{S}_n^k)]$ are undirected, we have $m_{u,v} = m_{v,u}$ for all appropriate u and v and thus \mathcal{A}_n^k is symmetric. Therefore we need only compute the upper right triangle of the matrix $\mathcal{A}_n^k(\ell)$. In fact in the following results, we compute the submatrices $A_{i,j}$ where $i \leq j$. This fully describes the matrix \mathcal{A}_n^k .

Theorem 2.1. *Let $\mathcal{A}_n^k = [A_{i,j}]_{1 \leq i, j \leq n+k}$, where $n \geq 3$ and $k \geq 0$. Assume $1 \leq i \leq j \leq n+k$, we have the following:*

1. *If $i = j$, then $A_{i,j} = [0]$, that is, a $(k+1) \times (k+1)$ matrix filled with zeros.*
2. *If $0 < j - i < n - 1$, then*

$$m_{u,v} = \begin{cases} 1 & \text{for } u \in \{i, i+1, \dots, j-1\} \text{ and } v \in \{i+k+1, i+k+2, \dots, j+k\} \\ 0 & \text{otherwise.} \end{cases}$$

3. *If $n-1 \leq j-i \leq n+k-1$, then*

$$m_{u,v} = \begin{cases} 1 & \text{for } u \in \{j+k+1, j+k+2, \dots, i+k\} \text{ and } v \in \{j, j+1, \dots, i-1\} \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry this completely describes the matrix \mathcal{A}_n^k .

Proof. As defined, we let $A_{i,j} = [m_{u,v}]$, where $u \in [i, i+1, \dots, i+k]_{\text{mod } (n+k)}$ and $v \in [j, j+1, \dots, j+k]_{\text{mod } (n+k)}$. Recall that $m_{u,v}$ denotes the number of strict alternating cycles between the dual of the critical pairs (b_i, a_u) and (b_j, a_v) . Since we are in the skeleton of the hypergraph there are no hyperedges. Hence we are only looking for edges which connect two vertices, namely sets of the form $\{(b, a), (b', a')\}$, where (a, b) and (a', b') are critical pairs of \mathbb{S}_n^k . By definition $\{(b, a), (b', a')\}$ is a strict alternating cycle if and only if the following conditions are satisfied:

1. $b || a$;
2. $a < b'$;
3. $b' || a'$; and
4. $a' < b$.

In each of the following cases, recall that for the value of $m_{u,v}$, we require $u \in [i, i+1, \dots, i+k]_{\text{mod } (n+k)}$ and $v \in [j, j+1, \dots, j+k]_{\text{mod } (n+k)}$. Given the definition of \mathbb{S}_n^k we have the following implications:

- Condition 1 implies $u \in \{i, i+1, \dots, i+k\}$.
- Condition 2 implies $u \in \{j-n+1, j-n+2, \dots, j-1\}$.

- Condition 3 implies $v \in \{j, j+1, \dots, j+k\}$.
- Condition 4 implies $v \in \{i-n+1, i-n+2, \dots, i-1\}$.

Thus, $m_{u,v}$ is nonzero whenever the above statements hold simultaneously and zero otherwise. We now consider the following cases.

Case 1: Assume that $i = j$. We claim that $A_{i,i} = [0]$. Suppose to the contrary that there exists a $u \in [i, i+1, \dots, k+i] \bmod (n+k)$ and $v \in [j, j+1, \dots, k+j] \bmod (n+k)$ such that $m_{u,v} \neq 0$. Then there exists an edge between the critical pairs (a_u, b_i) and (a_v, b_i) . This implies that $\{(b_i, a_u), (b_i, a_v)\}$ forms a strict alternating cycle. Condition (2) implies that $a_u < b_i$. This contradicts Condition (1) which states that $b_i \parallel a_u$. Therefore $A_{i,i}$ is the $(k+1) \times (k+1)$ zero matrix.

Case 2: If $0 < j-i < n-1$, then the above implications yield

$$m_{u,v} = \begin{cases} 1 & \text{for } u \in \{i, i+1, \dots, j-1\} \text{ and } v \in \{i-n+1, i-n+2, \dots, j+k\} \\ 0 & \text{otherwise.} \end{cases}$$

Case 3: If $n-1 \leq j-i \leq n+k-1$, then the above implications yield

$$m_{u,v} = \begin{cases} 1 & \text{for } u \in \{j+k+1, j+k+2, \dots, i+k\} \text{ and } v \in \{j, j+1, \dots, i-1\} \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 2.1. *Let $n \geq 3$ and $k \geq 0$, and without loss of generality, assume $j \geq i$. For $1 \leq i, j \leq n+k$ the following hold true:*

1. *If $0 \leq j-i < n-1$, then $A_{i,j}$ has $(j-i \bmod (n+k))^2$ non-zero entries.*
2. *If $n-1 \leq j-i \leq n+k-1$, then $A_{i,j}$ has $(i-j \bmod (n+k))^2$ non-zero entries.*

3 Adjacency matrix of $Sk[K^s(\rtimes_{\ell} \mathbb{S}_n^k)]$ when $n \geq k+3$

We begin by recalling the definition of layering introduced by Garcia and Silva in [1].

Definition 3.1. *Let $\mathbb{P}_1 = (X_1, P_1)$ and $\mathbb{P}_2 = (X_2, P_2)$ be two posets, such that there exists a bijection $\beta : \max(\mathbb{P}_1) \rightarrow \min(\mathbb{P}_2)$. The β -layering of \mathbb{P}_2 over \mathbb{P}_1 is a poset*

$$\mathbb{P}_1 \rtimes_{\beta} \mathbb{P}_2 = (X_1 \cup X_2, Q)$$

where Q is the transitive closure of

$$P_1 \cup P_2 \cup \{(x, \beta(x)), (\beta(x), x)\}_{x \in \max(\mathbb{P}_1)}.$$

In this process x is literally glued with $\beta(x)$. Since β is a bijection, there are no issues in doing so. In fact, this process can be repeated a finite number of times to obtain as many layers as desired.

Definition 3.2. The ℓ -layered generalized crown of \mathbb{S}_n^k is denoted by

$$\rtimes_{\ell} \mathbb{S}_n^k := \underbrace{\mathbb{S}_n^k \rtimes \cdots \rtimes \mathbb{S}_n^k}_{\ell \text{ times}}.$$

Notation 3.1. Elements in a layered generalized crown are denoted x_j^i , where the subscript of each element denotes its location within a row and the superscript denotes its row. When considering $\rtimes_{\ell} \mathbb{S}_n^k$ where $\ell \geq 2$, we let X^i denote the set of elements in the i^{th} row of $\rtimes_{\ell} \mathbb{S}_n^k$, namely $X^i = \{x_1^i, x_2^i, \dots, x_{n+k}^i\}$. Note that in this case $X = \cup_{i=1}^{\ell+1} X^i$.

Example 3.1. The following depicts two layers of \mathbb{S}_3^2 , that is $\mathbb{S}_3^2 \rtimes \mathbb{S}_3^2$.

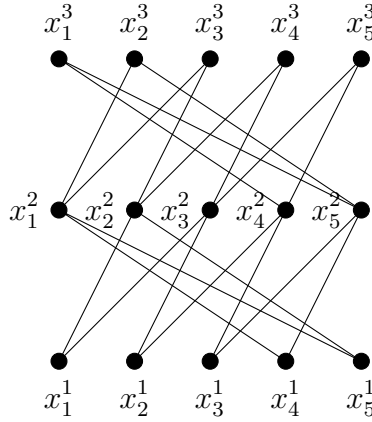


Figure 5: $\mathbb{S}_3^2 \rtimes \mathbb{S}_3^2$

Next we compute the adjacency matrix for the skeleton of the strict hypergraph of the ℓ -layered generalized crown \mathbb{S}_n^k . We denote this matrix by $\mathcal{A}_n^k(\ell)$. These adjacency matrices fall into one of three cases dependent upon the values of n, k and ℓ .

For the remainder of this section, we assume $n \geq k + 3$ and $\ell \geq 1$. In these crowns, critical pairs can only be produced from elements in adjacent rows; see [1, Lemma 4.1]. This occurs because n is large enough to compel an element of X^i to hit every element in X^{i-2} . To be precise, this means that for all $x_s^i \in X^i$ and any $x_r^{i-2} \in X^{i-2}$ we have $x_s^i \geq x_r^{i-2}$. Moreover for any $x_s^i \in X^i$ we have $x_s^i \geq x_r^t$ for all $t \leq i - 2$ and $1 \leq r \leq n + k$.

Notation 3.2. In order to distinguish between layers we introduce the following notation. For each $1 \leq r \leq \ell$, let $\mathbb{P}_r = \mathbb{S}_n^k$. When $1 \leq r \leq \ell$ and $1 \leq j \leq n + k$, we let $x_j^r \in \min(\mathbb{P}_r) = X^r$ and $x_j^{r+1} \in \min(\mathbb{P}_{r+1}) = \max(\mathbb{P}_r) = X^{r+1}$. When $r = \ell + 1$, we let $x_j^r \in \max(\mathbb{P}_{\ell}) = X^{\ell+1}$.

In the case where $n \geq k + 3$ and $\ell \geq 1$, the critical pairs of $\rtimes_{\ell} \mathbb{S}_n^k$ arise only through adjacent rows X^i and X^{i+1} . We order the critical pairs by lexicographical order on their

dual. Hence we label the first $(k+1)(n+k)$ rows/columns of the matrix $\mathcal{A}_n^k(\ell)$ as follows:

$$\begin{aligned}
& (x_1^1, x_1^2) \\
& (x_2^1, x_1^2) \\
& \vdots \\
& (x_{1+k}^1, x_1^2) \\
& (x_2^1, x_2^2) \\
& (x_3^1, x_2^2) \\
& \vdots \\
& (x_{2+k}^1, x_2^2) \\
& \vdots \\
& (x_i^1, x_i^2) \\
& \vdots \\
& (x_{i+k}^1, x_i^2) \\
& \vdots \\
& (x_{n+k}^1, x_{n+k}^2) \\
& \vdots \\
& (x_{(n+k)+k}^1, x_{n+k}^2)
\end{aligned} \tag{1}$$

The next $(k+1)(n+k)$ rows/columns are labeled similarly by considering the critical pairs created between X^2 and X^3 . In particular, the superscripts increase by one from those listed in (1). We continue this process and note that at the r^{th} step (when considering the critical pairs coming from X^r and X^{r+1}), where $1 \leq r \leq \ell$, we have the following row/column labels:

$$\begin{aligned}
& (x_1^r, x_1^{r+1}) \\
& \vdots \\
& (x_{1+k}^r, x_1^{r+1}) \\
& (x_2^r, x_2^{r+1}) \\
& \vdots \\
& (x_{2+k}^r, x_2^{r+1}) \\
& \vdots \\
& (x_{n+k}^r, x_{n+k}^{r+1}) \\
& \vdots \\
& (x_{(n+k)+k}^r, x_{n+k}^{r+1})
\end{aligned} \tag{2}$$

This process terminates when we consider $r = \ell$; namely when we obtain the critical pairs created from X^ℓ and $X^{\ell+1}$. This catalogs all the critical pairs that arise when $n \geq k+3$. This process yields a label for all of the rows/columns of the matrix $\mathcal{A}_n^k(\ell)$, which has dimensions $[(n+k)(k+1)\ell] \times [(n+k)(k+1)\ell]$.

To simplify the computation of matrix $\mathcal{A}_n^k(\ell)$, we decompose $\mathcal{A}_n^k(\ell)$ into ℓ^2 submatrices each of size $(k+1)(n+k) \times (k+1)(n+k)$ as described below. First, for any integer r , where $1 \leq r \leq \ell$, we let

$$L_r = \left[\begin{array}{cccccc} (x_1^r, x_1^{r+1}), (x_2^r, x_1^{r+1}), \dots, (x_{1+k}^r, x_1^{r+1}), (x_2^r, x_2^{r+1}), (x_3^r, x_2^{r+1}), \dots, (x_{2+k}^r, x_2^{r+1}), \\ \dots, (x_i^r, x_i^{r+1}), \dots, (x_{i+k}^r, x_i^{r+1}), \dots, (x_{n+k}^r, x_{n+k}^{r+1}), \dots, (x_{(n+k)+k}^r, x_{n+k}^{r+1}) \end{array} \right].$$

Notice L_r is an ordered list consisting of the labels of the rows/columns at the r^{th} step as given in (2). Now for any two integers i and j between 1 and ℓ , we let $A_{i,j}$ be the submatrix of $\mathcal{A}_n^k(\ell)$ whose rows are labeled by L_i and whose columns are labeled by L_j .

Therefore we write

$$\mathcal{A}_n^k(\ell) = [A_{i,j}]_{1 \leq i,j \leq \ell} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,j} & \cdots & A_{1,\ell} \\ A_{2,1} & \cdots & A_{2,j} & \cdots & A_{2,\ell} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i,1} & \cdots & A_{i,j} & \cdots & A_{i,\ell} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{\ell,1} & \cdots & A_{\ell,j} & \cdots & A_{\ell,\ell} \end{bmatrix}.$$

Next we describe the set of zero submatrices $A_{i,j}$ of $\mathcal{A}_n^k(\ell)$, when $n \geq k+3$ and $\ell \geq 1$.

Lemma 3.1. *Let $n \geq k+3$ and $\ell \geq 1$. If $A_{r,t}$ is a submatrix of $\mathcal{A}_n^k(\ell)$ where $|r-t| > 1$, then $A_{r,t} = [0]$.*

Proof. Suppose there exists a nonzero entry in $A_{r,t}$ where $|r-t| > 1$. This implies there exists a strict alternating cycle of the form

$$\{(x^{r+1}, x^r), (x^{t+1}, x^t)\},$$

where (x^r, x^{r+1}) and (x^t, x^{t+1}) are critical pairs in \mathbb{P}_r and \mathbb{P}_t , respectively. By definition of a strict alternating cycle, the following four conditions hold:

- (i) $x^r \parallel x^{r+1}$;
- (ii) $x^r \leq x^{t+1}$;
- (iii) $x^t \parallel x^{t+1}$; and
- (iv) $x^t \leq x^{r+1}$.

Since $|r-t| > 1$, without loss of generality we can assume that $r < t$.

Observe that condition (iv) gives a contradiction since $x^t \in \min(\mathbb{P}_t)$ and $x^{r+1} \in \max(\mathbb{P}_r)$ but the elements of $\min(\mathbb{P}_t)$ are above those of $\max(\mathbb{P}_r)$. \square

Lemma 3.2. *Let $n \geq k + 3$ and $\ell \geq 1$. If $A_{i,i+1}$ is a submatrix of $\mathcal{A}_n^k(\ell)$, then*

$$[A_{i,i+1}]_{(x^i, x_\alpha^{i+1}), (x_\beta^{i+1}, x^{i+2})} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise,} \end{cases}$$

where (x^i, x_α^{i+1}) and (x_β^{i+1}, x^{i+2}) are critical pairs of \mathbb{P}_i and \mathbb{P}_{i+1} , respectively.

Proof. Observe that $\{(x^{i+1}, x^i), (x^{i+2}, x^{i+1})\}$ is a strict alternating cycle provided that the following conditions are satisfied:

- (i) $x^{i+1} || x^i$
- (ii) $x^i \leq x^{i+2}$
- (iii) $x^{i+2} || x^{i+1}$
- (iv) $x^{i+1} \leq x^{i+1}$

Observe that Conditions (i) and (iii) hold by definition of critical pairs. Condition (ii) holds since $n \geq k + 3$. Now notice that the only comparable elements in $\min(\mathbb{P}_{i+1}) = \max(\mathbb{P}_i)$ are $x_\alpha^{i+1} = x_\beta^{i+1}$ provided $\alpha = \beta$. Namely if two elements in the same row are comparable, then they must be the same element. \square

Theorem 3.1. *If $n \geq k + 3$ and $\ell \geq 1$, then $\mathcal{A}_n^k(\ell) = [A_{i,j}]_{1 \leq i,j \leq \ell}$ where the submatrices $A_{i,j}$ are described as follows:*

- $A_{i,j} = [0]$ when $|i - j| \neq 1$; and
- $A_{i,j}$ is as described in Lemma 3.2 when $|i - j| = 1$.

Proof. Since $n \geq k + 3$, we can extend [1, Lemma 4.1] inductively to ℓ -layers to obtain that the set of critical pairs of $\bowtie_\ell \mathbb{S}_n^k$ is the disjoint union of the critical pairs of $\mathbb{P}_1, \dots, \mathbb{P}_\ell$. That is, $\text{crit}(\bowtie_\ell \mathbb{S}_n^k) = \text{crit}(\mathbb{P}_1) \sqcup \dots \sqcup \text{crit}(\mathbb{P}_\ell)$.

When critical pairs of a layered crown come solely from distinct layers we in fact can see that the adjacency matrix $\mathcal{A}_n^k(\ell)$ must be of the form described. Theorem 2.1 implies that $A_{i,i} = [0]$. Lemma 3.1 implies that $A_{i,j} = [0]$, when $|i - j| > 1$. Lastly, Lemma 3.2 yields the last condition. \square

4 Adjacency matrix of $Sk[K^s(\mathbb{S}_n^k)]$ when $3 \leq n < k + 3$

We break Section 4 into two subsections based upon the value of ℓ .

4.1 A small number of layers

In this case $1 \leq \ell \leq \left\lceil \frac{k+1}{n-2} \right\rceil$ and all critical pairs are formed from the extreme subposet; this is shown in the proof of [1, Theorem 4.3].

Theorem 4.1. *Let $3 \leq n < k+3$ and $1 \leq \ell \leq \left\lceil \frac{k+1}{n-2} \right\rceil$. Then $\mathcal{A}_n^k(\ell) = \mathcal{A}_{n+(\ell-1)(n-2)}^{k-(\ell-1)(n-2)}$.*

Proof. By [1, Theorem 4.3] the critical pairs of $\bowtie_\ell \mathbb{S}_n^k$ are exactly the critical pairs of the extreme subposet and $\mathcal{E}(\bowtie_\ell \mathbb{S}_n^k) \cong \mathbb{S}_{n+(\ell-1)(n-2)}^{k-(\ell-1)(n-2)}$, when $3 \leq n < k+3$ and $1 \leq \ell \leq \left\lceil \frac{k+1}{n-2} \right\rceil$. We apply Theorem 2.1 to fully describe the adjacency matrix of $Sk[K^s(\bowtie_\ell \mathbb{S}_n^k)]$. \square

4.2 A large number of layers

Now we consider $\ell > \left\lceil \frac{k+1}{n-2} \right\rceil$. The critical pairs of $\bowtie_\ell \mathbb{S}_n^k$ arise from various extreme subposets, as shown in the proof of [1, Theorem 4.4]. We begin by setting some notation.

Notation 4.1. *We adopt the notation used in the proof of [1, Theorem 4.3]. Let*

$$\mathbb{P} = \mathbb{P}_1 \bowtie \mathbb{P}_2 \bowtie \cdots \bowtie \mathbb{P}_\ell.$$

Setting $w = \left\lceil \frac{k+1}{n-2} \right\rceil$, we let $\mathcal{E}_j = \mathbb{P}(X^j \cup X^{j+w})$, where $j = 1, \dots, \ell - w + 1$.

From the proof of [1, Theorem 4.3] we know that the critical pairs of \mathbb{P} come from the incomparable elements in the subposet \mathcal{E}_j . Hence

$$\text{Crit}(\mathbb{P}) = \bigsqcup_{j=1}^{\ell-w+1} \text{Crit}(\mathcal{E}_j).$$

For $j = 1, \dots, \ell - w + 1$, let \mathcal{C}_j denote the elements of $\text{Crit}(\mathcal{E}_j)$ ordered using lexicographical order on the dual of the critical pairs of \mathcal{E}_j . For a fixed $j = 1, \dots, \ell - w + 1$, suppose $x_p^j \in X^j$ and $x_q^{j+w} \in X^{j+w}$ such that $x_p^j \parallel x_q^{j+w}$ in \mathbb{P} . Then the index q must be a value in the set of cyclic indexing values below:

$$q \in \{p + w(n-1) + 1, p + w(n-1) + 2, \dots, p + w - 1\}.$$

Therefore we can compute that for $j = 1, \dots, \ell - w + 1$,

$$\text{Crit}(\mathcal{E}_j) = \bigsqcup_{p=1}^{n+k} \left\{ (x_p^j, x_s^{j+w}) : s \in \{p + w(n-1) + 1, p + w(n-1) + 2, \dots, p + w - 1\} \right\}.$$

Let $A_{r,t}$ denote the submatrix of $\mathcal{A}_n^k(\ell)$ whose rows are labeled \mathcal{C}_r and the columns are labeled \mathcal{C}_t . Then $\mathcal{A}_n^k(\ell) = [A_{i,j}]_{1 \leq i,j \leq \ell-w+1}$.

Lemma 4.1. Let $3 \leq n < k + 3$ and $\ell > \left\lceil \frac{k+1}{n-2} \right\rceil$. If $A_{i,i}$ is a submatrix of $\mathcal{A}_n^k(\ell)$, then $A_{i,i} = \mathcal{A}_{n+(w-1)(n-2)}^{k-(w-1)(n-2)}$.

Proof. By [1, Theorem 4.3] we know that the critical pairs of $\bowtie_w \mathbb{S}_n^k$ are exactly the critical pairs of the extreme subposet $\mathcal{E}_i = (X^i \cup X^{i+w})$ and $\mathcal{E}_i = \mathbb{S}_{n+(w-1)(n-2)}^{k-(w-1)(n-2)}$. Applying Theorem 2.1 achieves the desired conclusion. \square

Lemma 4.2. Let $3 \leq n < k + 3$ and $\ell > \left\lceil \frac{k+1}{n-2} \right\rceil$. If $A_{i,j}$ is a submatrix of $\mathcal{A}_n^k(\ell)$ such that $0 < |i - j| = r < w$, then

$$[A_{i,j}]_{(x_\alpha^i, x_\beta^{i+w}), (x_\gamma^j, x_\delta^{j+w})} = \begin{cases} 1 & \text{if } \gamma \in [\beta - w + r, \beta - w + r - 1, \dots, \beta - (w - r)(n - 1)] \pmod{(n+k)} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Without loss of generality assume that $j > i$. Let $j = i + r$, where $r < w$. Assume $(x_\alpha^i, x_\beta^{i+w}) \in \text{Crit}(\mathcal{E}_i)$ and $(x_\gamma^j, x_\delta^{j+w}) \in \text{Crit}(\mathcal{E}_j)$. Then

$$\{(x_\beta^{i+w}, x_\alpha^i), (x_\delta^{j+w}, x_\gamma^j)\},$$

forms a strict alternating cycle if and only if the following two conditions are satisfied:

$$x_\alpha^i \leq x_\delta^{j+w}, \quad (3)$$

$$x_\gamma^j \leq x_\beta^{i+w}. \quad (4)$$

Using the fact that $j = i + r$, where $r < w$, Conditions (3) and (4) are (respectively) equivalent to

$$x_\alpha^i \leq x_\delta^{i+r+w} \quad (5)$$

$$x_\gamma^{i+r} \leq x_\beta^{i+w}. \quad (6)$$

Then Condition (5) holds since every element of X^i is comparable to every element of X^{i+r+w} . However, Condition (6) holds only when $x_\gamma^{i+r} \in D(x_\beta^{i+w})$. For $x_\beta^{i+w} \in X^{i+w}$ we compute the following information.

Downsets	Number of consecutively indexed elements
$D(x_\beta^{i+w}) \cap X^{i+w-1}$	$n - 1$
$D(x_\beta^{i+w}) \cap X^{i+w-2}$	$n - 1 + 1(n - 2)$
$D(x_\beta^{i+w}) \cap X^{i+w-3}$	$n - 1 + 2(n - 2)$
\vdots	\vdots
$D(x_\beta^{i+w}) \cap X^{i+r}$	$n - 1 + (w - r - 1)(n - 2)$

In fact

$$D(x_\beta^{i+w}) \cap X^{i+r} = \{x_{\beta+r-w}^{i+w}, x_{\beta+r-w-1}^{i+w}, x_{\beta+r-w-2}^{i+w}, \dots, x_{\beta-(w-r)(n-1)}^{i+w}\},$$

where the subscripts are taken modulo $(n+k)$. Therefore, $x_\gamma^{i+r} \in D(x_\beta^{i+w})$ if and only if

$$\gamma \in [\beta - w + r, \beta - w + r - 1, \dots, \beta - (w - r)(n - 1)] \pmod{(n+k)}.$$

□

Lemma 4.3. *Let $3 \leq n < k + 3$ and $\ell > \left\lceil \frac{k+1}{n-2} \right\rceil$. If $A_{i,j}$ is a submatrix of $\mathcal{A}_n^k(\ell)$ such that $|i - j| = w$, then*

$$[A_{i,j}]_{(x_\alpha^i, x_\beta^{i+w}), (x_\gamma^j, x_\delta^{j+w})} = \begin{cases} 1 & \text{if } \beta = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Without loss of generality assume that $j = i + w$. Assume $(x_\alpha^i, x_\beta^{i+w}) \in \text{Crit}(\mathcal{E}_i)$ and $(x_\gamma^j, x_\delta^{j+w}) \in \text{Crit}(\mathcal{E}_j)$. Then

$$\{(x_\beta^{i+w}, x_\alpha^i), (x_\delta^{j+w}, x_\gamma^j)\},$$

forms a strict alternating cycle if and only if the following two conditions are satisfied:

$$x_\alpha^i \leq x_\delta^{j+w}, \quad (7)$$

$$x_\gamma^j \leq x_\beta^{i+w}. \quad (8)$$

Using the fact that $j = i + w$, Conditions (7) and (8) are (respectively) equivalent to

$$x_\alpha^i \leq x_\delta^{i+2w} \quad (9)$$

$$x_\gamma^j \leq x_\beta^j. \quad (10)$$

Then Condition (9) holds since every element of X^i is hit by every element of X^{i+2w} . However Condition (10) only holds only when $\gamma = \beta$, as expected. □

Lemma 4.4. *Let $3 \leq n < k + 3$ and $\ell > \left\lceil \frac{k+1}{n-2} \right\rceil$. If $A_{i,j}$ is a submatrix of $\mathcal{A}_n^k(\ell)$ such that $|i - j| > w$, then $A_{i,j} = [0]$.*

Proof. Without loss of generality we assume that $j - i > w$ and let $(x^i, x^{i+w}) \in \text{Crit}(\mathcal{E}_i)$ and $(x^j, x^{j+w}) \in \text{Crit}(\mathcal{E}_j)$. Note that

$$\{(x^{i+w}, x^i), (x^{j+w}, x^j)\}$$

will never form a strict alternating cycle since $x^j \in X^j$ and $x^{i+w} \in X^{i+w}$, where $j > i + w$ therefore $x^j \not\leq x^{i+w}$. This implies that no strict alternating cycles exist, and thus $A_{i,j} = [0]$, whenever $|j - i| > w$. □

We now give the main result in this section.

Theorem 4.2. *If $3 \leq n < k + 3$ and $\ell > \left\lceil \frac{k+1}{n-2} \right\rceil$, then $\mathcal{A}_n^k(\ell) = [A_{i,j}]_{1 \leq i, j \leq \ell}$ where the submatrices $A_{i,j}$ are described as follows:*

- $A_{i,i}$ is as described in Lemma 4.1.
- $A_{i,j}$ is as described in Lemma 4.2 when $0 < |i - j| < w$.
- $A_{i,j}$ is as described in Lemma 4.3 when $|i - j| = w$.
- $A_{i,j} = [0]$ when $|i - j| > w$;

Proof. Follows from Lemmas 4.1, 4.2, 4.3, and 4.4. \square

We end this paper with an example that demonstrates how the matrix $\mathcal{A}_n^k(\ell)$ changes as more layers are added to a generalized crown.

Example 4.1. We compute $\mathcal{A}_3^1(3)$.

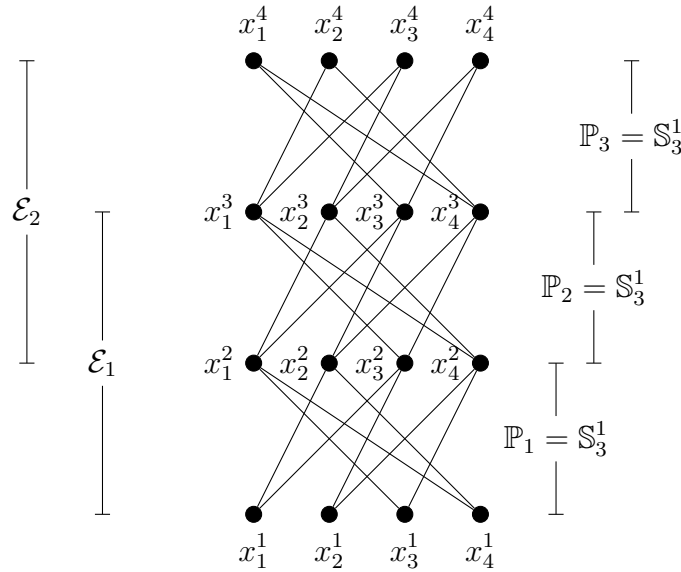


Figure 6: $\mathbb{S}_3^2 \rtimes \mathbb{S}_3^1$

In this case the critical pairs arise from the extreme subposets $\mathcal{E}_1 = \mathbb{P}(X^1 \cup X^3)$ and $\mathcal{E}_2 = \mathbb{P}(X^2 \cup X^4)$, as depicted in Figure 6. These critical pairs are as follows:

Critical pairs from \mathcal{E}_1 : $(x_1^1, x_2^3), (x_2^1, x_3^3), (x_3^1, x_4^3), (x_4^1, x_1^3)$
Critical pairs from \mathcal{E}_2 : $(x_1^2, x_4^4), (x_2^2, x_3^4), (x_3^2, x_4^4), (x_4^2, x_1^4)$.

The eight critical pairs imply that $\mathcal{A}_3^1(3)$ has dimension 8×8 . As before, we label the rows/columns by using lexicographical ordering on the dual of the critical pairs. That is, the rows and columns will be labeled as follows:

$$(x_4^1, x_1^3), (x_1^1, x_2^3), (x_2^1, x_3^3), (x_3^1, x_4^3), (x_4^2, x_1^4), (x_1^2, x_2^4), (x_2^2, x_3^4), (x_3^2, x_4^4).$$

To determine the non-zero entries of $\mathcal{A}_3^1(3)$ we compute from the above critical pairs the following strict alternating cycles:

$$\begin{aligned} &\{(x_1^3, x_4^1), (x_2^3, x_1^1)\}, \{(x_1^3, x_4^1), (x_3^3, x_2^1)\}, \{(x_1^3, x_4^1), (x_4^3, x_3^1)\}, \{(x_1^3, x_4^1), (x_1^4, x_2^2)\}, \{(x_1^3, x_4^1), (x_4^4, x_3^2)\}, \\ &\{(x_2^3, x_1^1), (x_3^3, x_2^1)\}, \{(x_2^3, x_1^1), (x_4^3, x_3^1)\}, \{(x_2^3, x_1^1), (x_1^4, x_2^2)\}, \{(x_2^3, x_1^1), (x_4^4, x_3^2)\}, \{(x_3^3, x_2^1), (x_4^3, x_3^1)\}, \\ &\{(x_3^3, x_2^1), (x_2^4, x_1^2)\}, \{(x_3^3, x_2^1), (x_4^4, x_3^2)\}, \{(x_4^3, x_3^1), (x_4^4, x_3^2)\}, \{(x_4^3, x_3^1), (x_4^4, x_3^2)\}, \{(x_1^4, x_2^2), (x_4^4, x_3^2)\}, \\ &\{(x_1^4, x_2^2), (x_4^4, x_3^2)\}, \{(x_1^4, x_2^2), (x_4^4, x_3^2)\}, \{(x_2^4, x_1^2), (x_3^4, x_2^2)\}, \{(x_2^4, x_1^2), (x_4^4, x_3^2)\}, \{(x_3^4, x_2^2), (x_4^4, x_3^2)\}. \end{aligned}$$

Therefore the 3-layered crown $\times_3 \mathbb{S}_3^1$ yields the matrix

$$\mathcal{A}_3^1(3) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

When we consider the 4-layered crown $\times_4 \mathbb{S}_3^1$, we obtain $\mathcal{A}_3^1(4)$, as shown in Table 1. When we consider the 5-layered crown $\times_5 \mathbb{S}_3^1$, we obtain $\mathcal{A}_3^1(5)$, as shown in Table 2.

0	1	1	1	1	0	0	1	0	1	0	0	0
1	0	1	1	1	1	1	0	0	0	0	1	0
1	1	0	1	1	0	1	1	0	0	0	0	1
1	1	1	0	1	0	0	1	1	1	0	0	0
1	1	0	0	0	0	1	1	1	1	0	0	1
0	1	1	1	0	1	0	1	1	1	1	0	0
0	0	1	1	1	1	1	0	1	0	1	1	0
1	0	0	1	1	1	1	1	0	0	0	1	1
0	0	0	1	1	1	0	0	0	0	1	1	1
1	0	0	0	0	0	1	1	0	1	0	1	1
0	1	0	0	0	0	0	1	1	1	1	0	1
0	0	1	0	0	1	0	0	1	1	1	1	0

Table 1: $\mathcal{A}_3^1(4)$

